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ON A GRAPH ASSOCIATED TO INVARIANT CONJUGACY CLASSES OF FINITE GROUPS

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ABSTRACT

Let A and G be finite groups of coprime orders such that A acts by automorphisms on G. We define the A-invariant conjugacy class graph of G to be the graph having as vertices the noncentral A-invariant conjugacy classes of G, and two vertices are connected by an edge if their cardinalities are not coprime. We prove that when the graph is disconnected then G is solvable.

1. Introduction

Let A and G be finite groups and suppose that A acts by automorphisms on G. Then A acts on the set of conjugacy classes of G and we will consider those classes which are fixed by A. We define the invariant conjugacy class graph $\Gamma_A(G)$ as follows: the vertices of $\Gamma_A(G)$ are the noncentral A-invariant conjugacy classes of G, and two vertices are connected by an edge if their cardinalities are not coprime. When $A = 1$ or the action is trivial, then $\Gamma_A(G)$ is just the graph $\Gamma(G)$ defined in [3].

During the last years many properties related to $\Gamma(G)$ have been studied. It was proved in Theorem 1 of [3] that $\Gamma(G)$ has at most two connected components, and it is easy to see that this result and the same proof given there hold for the graph $\Gamma_A(G)$.

It was also proved in [3] that the diameter of $\Gamma(G)$, $d(\Gamma(G))$, is upper bounded by 4 when $\Gamma(G)$ is a connected graph. It was proved later in [4] and in [1] that in fact 3 is the best bound for $d(\Gamma(G))$. It is worth mentioning that far-reaching

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research appears in [6] which studies a generalization of the graph $\Gamma(G)$. The same proof of Theorem 4 of [3], but now considering the action of A on G and invariant conjugacy classes of G, works to show that the diameter of $\Gamma_A(G)$, $d(\Gamma_A(G))$, is less than or equal to 4.

Now consider the case where the graph $\Gamma(G)$ is not connected. It was first proved in [7], and later reproved in [3] and [1], that this condition is equivalent to the fact that $G/Z(G)$ is a Frobenius group where the inverse images in G of the kernel and a complement are abelian. Consequently, each of the two components in $\Gamma(G)$ is a complete graph. Now, if we assume that A acts on G and $\Gamma_A(G)$ is disconnected, an upper bound for the diameter of each component of $\Gamma_A(G)$ can easily be obtained, by using the same methods as in [3].

Although the groups with $\Gamma(G)$ disconnected are completely determined, we do not know how to characterize the action on finite groups with $\Gamma_A(G)$ disconnected. However, if we assume that $|A|$ and $|G|$ are coprime, our main result obtains the solvability of G from a result of [2], which uses the classification of finite simple groups.

THEOREM A: *Suppose that a group A acts coprimely on a group G and that $\Gamma_A(G)$ has exactly two connected components. Then G is solvable.*

2. Proofs

The following lemma plays an important role in the proof of the theorems, and it is an observation that Lemmas 1 and 2 of [3] hold under the group action hypothesis. If B and C are vertices of $\Gamma_A(G)$, the distance will be denoted by $d(B,C)$.

LEMMA 1: *Suppose that a group A acts on a finite group G and choose $B = \mathrm{Cl}_G(b)$ and $C = \mathrm{Cl}_G(c)$ two A-invariant conjugacy classes of G. If $(|B|,|C|) = 1$, then the following properties hold:*

(a) $C_G(b)C_G(c) = G$.

(b) *BC is an A-invariant conjugacy class of G and $BC = CB$.*

(c) *Suppose that $d(B,C) \geq 3$ (allowing $d(B,C) = \infty$) and $|B| < |C|$. Then BC is an A-invariant conjugacy class of G, $|BC| = |C|$, $CBB^{-1} = C$, $C\langle BB^{-1}\rangle = C$, $\langle BB^{-1}\rangle \subseteq \langle CC^{-1}\rangle$ and $|\langle BB^{-1}\rangle|$ divides $|C|$.*

Proof: It suffices to mimic the proofs of Lemmas 1 and 2 of [3] with the additional hypothesis of the group action of A on G. ∎

THEOREM 1: *Suppose that a group A acts on a group G. Then $\Gamma_A(G)$ has at most two connected components.*

Proof: Mimic the proof of Theorem 1 of [3], but considering the group action hypothesis and Lemma 1. ∎

We summarize the results for the diameter of the invariant conjugacy class graph in the following theorem.

THEOREM 2: *Suppose that a group A acts on a group G.*
(a) *If $\Gamma_A(G)$ is connected, then $d(\Gamma_A(G)) \leq 4$.*
(b) *If $\Gamma_A(G)$ is not connected, the diameter of each connected component is at most 2.*

Proof: In order to obtain (a), it suffices to reproduce the proof of Theorem 4 of [3] with the additional hypothesis of coprime action and Lemma 1.

For proving (b), suppose that $\Gamma_A(G)$ is not connected but $d(\Gamma_A(G)) \geq 3$. Then, we may choose B and C belonging to the same connected component and such that $d(B,C) = 3$. Without loss, suppose that $|B| < |C|$, so by Lemma 1(c), $|\langle BB^{-1}\rangle| \mid |C|$, and choose D to belong to the other connected component of $\Gamma_A(G)$. Suppose first that $|D| < |B|$ and, again by Lemma 1(c), $|\langle DD^{-1}\rangle|$ divides $|B|$ and $\langle DD^{-1}\rangle \subseteq \langle BB^{-1}\rangle$. Therefore, we conclude that $|\langle DD^{-1}\rangle|$ divides both $|B|$ and $|C|$, contradicting $d(B,C) = 3$. Suppose now that $|B| < |D|$. Then $|\langle BB^{-1}\rangle|$ divides $|D|$ (and $|C|$), which is a contradiction. Thus, (a) is proved. ∎

LEMMA 2: *Suppose that G is a finite group. If the length of each conjugacy class is not divisible by r, then G has a central Sylow r-subgroup.*

Proof: See Theorem (33.4) of [5]. ∎

In order to prove Theorem A we also need some facts on coprime action. The first one is a well-known result.

LEMMA 3: *Let A act coprimely on G. Then $B \to B \cap \mathbf{C}_G(A)$ defines a bijection from the set of A-invariant conjugacy classes of G onto the set of conjugacy classes of $\mathbf{C}_G(A)$. Furthermore, $|B \cap \mathbf{C}_G(A)|$ divides $|B|$ for any A-invariant class B of G.*

Proof: The first assertion is just Corollary (14.4) of [5].

Let B be an A-invariant class of G and write $C = \mathbf{C}_G(A)$. We prove that $|B \cap C|$ divides $|B|$. Notice that there exists some $c \in B \cap C$, whence $B = \mathrm{Cl}_G(c)$. Then $|B| = |G : C_G(c)|$ and $|B \cap C| = |\,\mathrm{Cl}_C(c)| = |C : C_C(c)|$. Let p be a prime divisor of $|C : C_C(c)|$. Observe that $C_G(c)$ is an A-invariant subgroup. Since any A-invariant p-subgroup is contained in an A-invariant Sylow p-subgroup, we may choose an A-invariant Sylow p-subgroup P of G such that $P \cap C_G(c)$ is an A-invariant Sylow p-subgroup of $C_G(c)$. Also, it is well known that $P \cap C$ is a Sylow p-subgroup of C, and similarly, $P \cap C \cap C_G(c)$ is a Sylow subgroup of $C \cap C_G(c)$. Therefore,

$$|C : C \cap C_G(c)|_p = |C|_p / |C \cap C_G(c)|_p = |P \cap C : P \cap C \cap C_G(c)|.$$

But $|P \cap C : P \cap C \cap C_G(c)| \le |P : P \cap C_G(c)| = |G|_p / |C_G(c)|_p = |G : C_G(c)|_p$, so we conclude that $|\,\mathrm{Cl}_C(c)|_p \le |\,\mathrm{Cl}_G(c)|_p$ for all primes p, and we are done. ∎

LEMMA 4: *Suppose that A acts coprimely on G and that N is an A-invariant normal subgroup of G. Then, for any A-invariant conjugacy class $\mathrm{Cl}_{G/N}(gN)$ of G/N, there exists an A-invariant class $\mathrm{Cl}_G(c)$ of G such that $|\,\mathrm{Cl}_{G/N}(gN)|$ divides $|\,\mathrm{Cl}_G(c)|$.*

Proof: Let $\mathrm{Cl}_{G/N}(gN)$ be an A-invariant class of G/N and write $C = \mathbf{C}_G(A)$. By coprime action $\mathbf{C}_{G/N}(A) = CN/N$, so by Lemma 3 there exists some $cN \in CN/N$ such that $\mathrm{Cl}_{G/N}(gN) = \mathrm{Cl}_{G/N}(cN)$. It is clear that $\mathrm{Cl}_G(c)$ is A-invariant and $|\,\mathrm{Cl}_{G/N}(cN)|$ divides $|\,\mathrm{Cl}_G(c)|$, as required. ∎

The following is a key fact for proving Theorem A and it relies on the classification of finite simple groups.

THEOREM 3: *Suppose that A acts coprimely on G. If $\mathbf{C}_G(A)$ is nilpotent then G is solvable.*

Proof: This is Theorem B of [2]. ∎

Proof of Theorem A: Let G be any group with a disconnected invariant graph, and let X_1 and X_2 be its components. We will prove that G is solvable by induction on $|G|$. We denote by π_1 and π_2 the sets of prime divisors of the lengths of the classes in X_1 and X_2, respectively. Certainly $\pi_1 \cap \pi_2 = \emptyset$ and classes in distinct components satisfy the hypothesis of Lemma 1(c).

Let D be an A-invariant conjugacy class of G with the largest length. Without loss of generality we assume that $D \in X_2$. Set $M = \langle B | B \in X_1 \rangle$ and

$N = \langle BB^{-1} | B \in X_1 \rangle$. Clearly M and N are non-identity A-invariant normal subgroups of G and $N = [M, G]$. We proceed with a series of steps.

STEP 1: M is a proper subgroup of G.

Proof: Let $\mathcal{D}$ be the union of all the A-invariant conjugacy classes of G of maximal size. Obviously, these classes are in X_2. If D is one of them and $B \in X_1$, then BD is an A-invariant conjugacy class of G and $|BD| = |D|$. Thus, $BD \subseteq \mathcal{D}$, so we conclude that $B\mathcal{D} \subseteq \mathcal{D}$ and by cardinalities $B\mathcal{D} = \mathcal{D}$. Moreover, $M\mathcal{D} = \mathcal{D}$. Hence, $\mathcal{D}$ is the union of cosets of the normal subgroup M. In particular $|M|$ divides $|\mathcal{D}|$ and M is proper in G.

STEP 2: There exists an A-invariant conjugacy class of G contained in $G - M$ and N is a π_2-group.

Proof: Since $|M|$ divides $|\mathcal{D}|$, but it is not possible that $M = \mathcal{D}$, it follows that $(G - M) \cap \mathcal{D} \neq \emptyset$. Therefore, as $G - M$ is the union of conjugacy classes, there exists some $D \in \mathcal{D}$ such that $D \subseteq G - M$. For any $B \in X_1$ notice that BD is an A-invariant conjugacy class which satisfies $|BD| = |D|$, $DBB^{-1} = D$, whence $DN = D$. Thus, $|N|$ divides $|D|$, which is only divisible by primes in π_2. Therefore, N is a π_2-group.

STEP 3: M is abelian.

Proof: We first prove that $N \subseteq Z(M)$. Let $B \in X_1$, $b \in B$. Then $|N : C_N(b)|$ certainly divides both $|N|$ and $|G : C_G(b)| = |B|$. By Step 2, $(|N|, |B|) = 1$. Thus, $N = C_N(b)$ and since M is generated by all classes $B \in X_1$, it follows that N is central in M, as wanted. Now, since $[M, G] = N$, we deduce that M is nilpotent. Write $M = R \times Z$, where Z is the largest Hall subgroup of M which is contained in $Z(G)$. Let q be any prime dividing $|R|$, and choose Q a Sylow q-subgroup of R. Then Q is normal in G and $N = [M, G] \supseteq [R, G] \supseteq [Q, G]$. Since Q is not central in G, we deduce that q must divide $|N|$. Therefore, $\pi(R) \subseteq \pi(N) \subseteq \pi_2$. Arguing as at the beginning of this step, it follows that $R \subseteq Z(M)$, and consequently M is abelian.

STEP 4: G has a unique solvable minimal normal A-invariant subgroup. Consequently, M is a p-group for some prime $p \in \pi_2$.

Proof: We may choose $B \in X_1$ and $C \in X_2$ such that $|B| < |C|$. We know that $\langle BB^{-1} \rangle \subseteq \langle CC^{-1} \rangle$. Suppose that there exists a solvable minimal A-invariant normal subgroup of G, say K, such that $K \neq \langle BB^{-1} \rangle$. This implies that $bK \notin Z(G/K)$ for some $b \in B$, since in other case $\langle BB^{-1} \rangle \subseteq K$, and the minimality of K yields the equality, a contradiction. Similarly, we obtain $cK \notin Z(G/K)$ for some $c \in C$. Now, if $\Gamma_A(G/K)$ is not connected, we have by induction that G/K is solvable, and the proof is finished. Therefore, we may assume that $\Gamma_A(G/K)$ is connected, so $\mathrm{Cl}_{G/K}(bK)$ and $\mathrm{Cl}_{G/K}(cK)$ are noncentral A-invariant classes which must be connected in $\Gamma_A(G/K)$. Since $|\mathrm{Cl}_{G/K}(bK)|$ divides $|\mathrm{Cl}_G(b)| = |B|$ and $|\mathrm{Cl}_{G/K}(cK)|$ divides $|\mathrm{Cl}_G(c)| = |C|$, it easily follows, by using Lemma 4, that B and C are connected in $\Gamma_A(G)$, which is a contradiction. Therefore, G possesses a unique solvable minimal A-invariant normal subgroup (which is equal to $\langle BB^{-1}\rangle$). The consequence in the statement of this step follows from the fact that M is abelian and N is a π_2-group.

Step 5: $Z(G) \cap \mathbf{C}_G(A) \subseteq \mathbf{C}_M(A)$.

Proof: If $z \in Z(G) \cap \mathbf{C}_G(A)$ and $B \in X_1$, then clearly Bz is an A-invariant class of G, with the same size as B. Thus $B, Bz \subseteq M$ and this implies $z \in \mathbf{C}_M(A)$.

Step 6: For any $B \in X_1$ and $b \in B \cap \mathbf{C}_G(A)$, we have that $\mathbf{C}_{C_G(b)}(A)$ is a p-group.

Proof: Choose $B \in X_1$ and $b \in B \cap \mathbf{C}_A(G)$. Notice that $C_G(b)$ is an A-invariant subgroup containing M, since M is abelian by Step 3. Then $\mathbf{C}_M(A) \subseteq \mathbf{C}_{C_G(b)}(A)$, and if equality holds then there is nothing to prove. Thus, we may assume that there exists some $c \in \mathbf{C}_{C_G(b)}(A) - \mathbf{C}_M(A)$. By Step 5, we have $Z(G) \cap \mathbf{C}_G(A) \subseteq \mathbf{C}_M(A)$. This implies that $\mathrm{Cl}_G(c)$ is a noncentral A-invariant class of G outside M, so trivially it belongs to X_2. Also, by Step 4, we know that b is a p-element and we write $c = c_p c_{p'}$, with $c_p, c_{p'} \in \mathbf{C}_{C_G(b)}(A)$, c_p and $c_{p'}$ being the p-part and p'-part of c, respectively. Since b and $c_{p'}$ commute and have coprime order, then $C_G(bc_{p'}) = C_G(b) \cap C_G(c_{p'})$ and consequently both $|B|$ and $|\mathrm{Cl}_G(c_{p'})|$ divide $|\mathrm{Cl}_G(bc_{p'})|$. Similarly, but now reasoning with c_p and $c_{p'}$, we obtain that $|\mathrm{Cl}_G(c_{p'})|$ also divides $|\mathrm{Cl}_G(c)|$, which is a π_2-number. It follows that if $|\mathrm{Cl}_G(c_{p'})| > 1$, then $|\mathrm{Cl}_G(bc_{p'})|$ is divisible by primes in π_1 and π_2 at the same time, but this is not possible. Therefore, $|\mathrm{Cl}_G(c_{p'})| = 1$, that is, $c_{p'} \in Z(G) \cap \mathbf{C}_G(A) \subseteq \mathbf{C}_M(A)$. As $\mathbf{C}_M(A)$ is a p-group, then $c_{p'} = 1$, so c is a p-element and consequently $\mathbf{C}_{C_G(b)}(A)$ is a p-group.

Step 7: $\mathbf{C}_G(A)$ is a $(\pi_1 \cup \{p\})$-group.

Proof: Suppose that there exists some prime r, distinct from p, dividing $|\mathbf{C}_G(A)|$. Choose $b \in B \cap \mathbf{C}_G(A)$ for some $B \in X_1$. If r does not divide $|B| = |G : C_G(b)|$, then there exists some A-invariant Sylow r-subgroup R of G such that $R \subseteq C_G(b)$. Then, $R \cap \mathbf{C}_G(A)$ is a nontrivial Sylow r-subgroup of $\mathbf{C}_G(A)$, which trivially lies in $C_G(b)$. This contradicts Step 6 and therefore any such prime r must divide $|B|$, so r is a π_1-number and the claim is proved.

STEP 8: We may asssume that $Z(\mathbf{C}_G(A)) \subseteq \mathbf{C}_M(A)$.

Suppose first that the size of any conjugacy class of $\mathbf{C}_G(A)$ is a p-number. By Lemma 2 and Step 7, it follows that $\mathbf{C}_G(A)$ has a central π_1-Hall subgroup, and we deduce that $\mathbf{C}_G(A)$ is nilpotent. Therefore, by Theorem 3, G is solvable and the proof is finished.

Thus, we can assume that there exists a conjugacy class in $\mathbf{C}_G(A)$, say C, such that $|C| > 1$ is a π_1 number. Furthermore, by Lemma 3, we can write $C = B \cap \mathbf{C}_G(A)$, for some A-invariant class B of G. Also, since $|C|$ divides $|B|$, then $|B|$ must be a π_1-number too, so $B \subseteq M$.

Now, if $z \in Z(\mathbf{C}_G(A))$, then certainly Cz is a conjugacy class of $\mathbf{C}_G(A)$ with the same size as C. Therefore, if $c \in C$ then $\mathrm{Cl}_G(cz)$ is an A-invariant conjugacy class of G. By applying Lemma 3, we obtain $\mathrm{Cl}_G(cz) \cap \mathbf{C}_G(A) = Cz$, and furthermore $|C| = |Cz|$ divides $|\mathrm{Cl}_G(cz)|$. Then $|\mathrm{Cl}_G(cz)|$ is a π_1-number too. In particular, $cz \in M$, and since $c \in M$, we obtain that $z \in M$.

STEP 9: $\Gamma(\mathbf{C}_G(A))$ is a disconnected graph and has classes whose size is a π_1-number and classes whose sizes are p-numbers.

Proof: We will prove first that $\mathbf{C}_G(A)$ has classes whose size is divisible by p and classes whose size is divisible by primes in π_1. Suppose that p does not divide any conjugacy class length in $\mathbf{C}_G(A)$. Then, by Lemma 2, $\mathbf{C}_G(A)$ has a central Sylow p-subgroup. Since M is a p-group, it follows that $M \cap \mathbf{C}_G(A) \subseteq Z(\mathbf{C}_G(A))$ (in fact, the equality holds by Step 5). Therefore, if we choose $b \in B \cap \mathbf{C}_G(A)$ for some $B \in X_1$, we obtain $\mathbf{C}_G(A) \subseteq C_G(b)$. By Step 6, we deduce that $\mathbf{C}_G(A)$ is a p-group, and applying Theorem 3 we conclude that G is solvable and our theorem is proved. Therefore, p must divide some conjugacy class length in $\mathbf{C}_G(A)$.

On the other hand, since we can assume that $\mathbf{C}_G(A)$ is not a p-group, we choose a prime q dividing $|\mathbf{C}_G(A)|$, distinct from p. Notice that $q \in \pi_1$, by Step 7. If q does not divide any conjugacy class length in $\mathbf{C}_G(A)$, then, again by Lemma 2, $\mathbf{C}_G(A)$ has a nontrivial central Sylow q-subgroup. This is not possible because $Z(\mathbf{C}_G(A))$ is a p-group by Steps 4 and 8. Thus, $\mathbf{C}_G(A)$ has classes whose size is divisible by primes in π_1, so our assertion is proved.

Now, by Lemma 3, any size class in $\mathbf{C}_G(A)$ divides the size of some A-invariant class in G. These facts allow us to conclude that the class sizes in $\mathbf{C}_G(A)$ are just π_1-numbers or p-numbers, and that $\Gamma(\mathbf{C}_G(A))$ is disconnected.

STEP 10: Conclusion.

Proof: Write $C = \mathbf{C}_G(A)$. The groups with a disconnected conjugacy class graph are completely characterized by Theorem 2 of [3]. By applying this result to C, we have that $C/Z(C)$ is a Frobenius group, and if $K/Z(C)$ and $L/Z(C)$ denote the kernel and the complement of $C/Z(C)$, respectively, then both K and L are abelian. Now, we consider $M \cap C/Z(C)$, which is a normal subgroup of $C/Z(C)$. It is well known that any normal subgroup in a Frobenius group contains or is contained in its kernel. Thus, we have two possibilities: $M \cap C \subset K$ or $K \subseteq M \cap C$.

Suppose first that $M \cap C \subset K$ and take $k \in K - M \cap C$. As $C/Z(C)$ is a Frobenius group and K and L are abelian, it follows that the class sizes in $C/Z(C)$ are exactly $\{1, |K/Z(C)|, |L/Z(C)|\}$. Moreover, we have $1 \neq |\operatorname{Cl}_{C/Z(C)}(kZ(C))| = |L/Z(C)|$, since $k \notin Z(C)$ and the centre of a Frobenius group is trivial.

We know by Step 8 that $Z(C) \subseteq M \cap C$ and we see now that this containment is proper. By Step 9, there exists a noncentral conjugacy class of C, say E, whose size is a π_1-number. Also, $E = B \cap C$ for some $B \in X_1$, by Lemma 3. Then, $1 \neq \langle E \rangle \subseteq \langle B \rangle \cap C \subseteq M \cap C$, $\langle E \rangle$ being trivially noncentral in C. Thus, $M \cap C$ contains properly $Z(C)$, as wanted.

Therefore, $p \mid |M \cap C/Z(C)| \mid |K/Z(C)|$. Since $(|K/Z(C)|, |L/Z(C)|) = 1$, by Step 7 we conclude that $|L/Z(C)|$ is a π_1-number. Now, $|\operatorname{Cl}_{C/Z(C)}(kZ(C))| = |L/Z(C)|$ certainly divides $|\operatorname{Cl}_C(k)|$, and by Lemma 3, $|\operatorname{Cl}_C(k)|$ divides $|\operatorname{Cl}_G(k)|$. It follows that $|\operatorname{Cl}_G(k)|$ must be a π_1-number too, so $k \in M \cap C$, which is a contradiction. This case is finished.

Suppose now that $K \subseteq M \cap C$. Since $C/K \cong L/Z(C)$ is abelian, we have that

$$C/M \cap C \cong CM/M = \mathbf{C}_{G/M}(A)$$

is abelian too. By applying Theorem 3, we obtain that G/M is solvable. Since M is abelian, then G is solvable and the proof is finished. ∎

Suppose that A acts coprimely on G. The proof of the solvability of G when $\Gamma_A(G)$ is disconnected has been reduced to the case in which $\Gamma(\mathbf{C}_G(A))$ is diconnected too. It could be thought, considering Lemma 3, that there exists some kind of correspondence between the components of $\Gamma_A(G)$ and the components of $\Gamma(\mathbf{C}_G(A))$. The following example shows that $\Gamma_A(G)$ may be connected, whereas $\Gamma(\mathbf{C}_G(A))$ is a disconnected graph.

Example: Let $G = PSL(2, 2^5)$, the projective special linear group on the field F of 2^5 elements, which has order $2^5.3.11.31$. Let $A = \langle \sigma \rangle$, where σ is the automorphism of order 5 of F defined by $u^\sigma = u^2$ for any $u \in F$. Then, A acts coprimely on G. As σ fixes only the prime field of F, it is easy to see that

$\mathbf{C}_G(A) \cong PSL(2,2) \cong S_3$. From the first step of the proof of our Theorem A, it follows (without making use of the classification of finite simple groups) that the invariant conjugacy class graph of a simple group must be connected, and thus, so is $\Gamma_A(G)$. (In fact, it can be proved, but using a result which relies on the classification, that $\Gamma_A(G)$ is a complete graph as in Proposition 5 of [3].) However, $\Gamma(\mathbf{C}_G(A))$ is a disconnected graph, since the noncentral class sizes in $\mathbf{C}_G(A)$ are $\{2,3\}$. ∎

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